Northwestern Ontario Annual High School Math Contest

(1) What is the minimum length of a sequence of consecutive integers such that their product is guaranteed to be divisible by 2022?

Solution: Since $2022 = 2 \times 3 \times 337$, any sequence of consecutive integers of length 337 must contain a multiple of 2, a multiple of 3, and a multiple of 337. Therefore, the product of all the consecutive integers in such a sequence is divisible by 2022. On the other hand, we note that the sequence $1, 2, \cdots, 336$, which has 336 consecutive integers, does not contain any multiple of 337, hence the product $1 \times 2 \times \cdots \times 336$ is not divisible by 2022. The answer is thus 337.

(2) Given real numbers r and s such that $r + s = 1$, $r^4 + s^4 = 7$, find $r^5 + s^5 + 5rs$.

Solution: Note that

$$\begin{aligned} 1 = (r+s)^4 &= r^4 + 4r^3s + 6r^2s^2 + 4rs^3 + s^4 \\ &= r^4 + s^4 + 4rs(r+s)^2 - 2r^2s^2 = 7 + 2rs(2 - rs). \end{aligned}$$

Thus $y = rs$ is solution of $y^2 - 2y = 3$, i.e. $y = rs = 3$ or $y = rs = -1$. We exclude the case $rs = 3$, since this would lead to

$$1 = (r+s)^2 = r^2 + s^2 + 2rs = r^2 + s^2 + 6,$$

which is a contradiction. Thus $rs = -1$. Then,

$$\begin{aligned} 7 = (r+s)(r^4+s^4) = r^5 + s^5 + rs^4 + r^4s &= r^5 + s^5 + rs(r^3 + s^3) \\ &= r^5 + s^5 + rs(r+s)(r^2 - rs + s^2) = r^5 + s^5 + rs[(r+s)^2 - 3rs] \\ &= r^5 + s^5 + rs[1 - 3rs] = r^5 + s^5 + 5rs - rs[4 + 3rs] \\ &= r^5 + s^5 + 5rs + 1, \end{aligned}$$

hence $r^5 + s^5 + 5rs = 6$.

(3) Consider the set of integer numbers $\{94 \leq k \leq 188\}$. For each such k, let $d(k)$ be its greatest odd divisor. Find the sum $d(94) + d(95) + \cdots + d(188)$.

Solution: Let $A_n := \{k : n < k \leq 2n\}$, $n = 94$. Note first that $k \geq d(k)$ for any integer k. Moreover, if two integers, say $m, l \in A_n$ have the same greatest odd divisor, then one is a power of two times the other, which is not possible. Thus $d(k)$, $n < k \leq 2n$, are all distinct and do not exceed $2n - 1$, and the only possibility is thus

$$\{d(k) : n < k \leq 2n\} = \{1, 3, 5, \cdots, 2n-1\}.$$

Since $d(94) = 47$, we infer that

$$d(94) + d(95) + \cdots + d(188) = 47 + 1 + 3 + 5 + \cdots + 187 = 8883.$$

(4) Next year, on her birthday, Alice's age will be the sum of the digits of the year she was born. What are her possible birth years?

Solution: The sum of all digits of a natural number not exceeding 2023 is at most 28, which is reached at the number 1999: if a number less than 10000 were to have a sum of its digits exceeding 28, then it would have to increase its leftmost digit to at least 2, but the only number less than 3000 with sum of digits exceeding 28 is 2999. Increasing the leftmost digit to at least 3 would make it automatically larger than 2023. Thus Alice is born on or after 1995. It is thus straightforward to check that only 1997 and 2015 are acceptable answers.

(5) In how many ways can we write 2022 as a sum of a sequence of two or more non decreasing positive integers, such that the difference between the largest and smallest term is at most 1?

Solution: Write $2022 = dq + r = q(d-r) + r(q+1)$, $0 \leq r < d$. Thus 2022 can be written as a sum of $(d-r)$ copies of the value q, plus r copies of the value $q+1$. The case $d = 2022$ corresponds to writing 2022 as a sum of 2022 copies of 1. The case $d = 1$ corresponds to writing 2022 as a sum of 2022 itself, which is not acceptable. Thus all choices $1 < d \leq 2022$ are acceptable, for a total of 2021 ways.

(6) How many ordered pairs (a, b) of integers are there that satisfy the equation $a^2 + b^2 + 2a + 4b = 8$?

Solution: Rewriting $a^2 + b^2 + 2a + 4b = 8$ as $(a^2 + 2a + 1) + (b^2 + 4b + 4) = 8 + 1 + 4$ we get $(a+1)^2 + (b+2)^2 = 13$. The only way to write 13 as a sum of two squares is $13 = 9 + 4$ If $(a+1)^2 = 9$ and $(b+2)^2 = 4$, we get $a = 2$ or $a = -4$, and $b = 0$ or $b = -4$. These possibilities give us the ordered pairs $(2, 0), (2, -4), (-4, 0)$, and $(-4, -4)$. If $(a+1)^2 = 4$ and $(b+2)^2 = 9$, we get $a = 1$ or $a = -3$, and $b = 1$ or $b = -5$. These possibilities give us the ordered pairs $(1, 1), (1, -5), (-3, 1)$, and $(-3, -5)$. There are therefore 8 possible pairs.

(7) We have three discs of radius 1 in the plane arranged so that the boundary circle of each disc passes through the centres of the other two discs. What is the area of the intersection of the three discs?

Solution: Let A, B, and C, denote the centres of the three circles. The triangle with vertices A, B, and C is an equilateral triangle with side length 1, so its area is $(1/2)(1)(\sin(\pi/3)) = \sqrt{3}/4$. Call this triangle T. Let S_C denote the wedge of the disc with centre C that is bounded by the line segments CA and CB and the shorter arc of the circle centred at C going from A to B. Then S_C has area 1/6 that of the disc centred at C, so its area is $\pi/6$. Define the wedges S_A and S_B similarly. The intersection of the three discs is the union of S_A, S_B, and S_C, and the intersection of any two of the wedges is T. Thus the area of the intersection of the three discs is $3(\pi/6) - 2(\sqrt{3}/4) = (\pi - \sqrt{3})/2$.

(8) A square S_1 has side length 1. A circle C_1 is inscribed inside S_1, and then another square S_2 is inscribed inside C_1. This process is continued, giving two sequences, S_n and C_n. What is the area of square S_8?

Solution: The side length of square S_1 is 1. the diameter of circle C_1 is the side length of S_1, so this is 1. The diagonal of square S_2 is the diameter of circle C_1, so this is 1, and the side length of square S_2 is $1/\sqrt{2}$. Continuing in this way, we see that the side length of square S_{n+1} is $1/\sqrt{2}$ times that of square S_n for $n \geq 1$. Thus the side length of S_8 is $(1/\sqrt{2})^7$ and its area is $((1/\sqrt{2})^7)^2 = 1/2^7 = 1/128$.

(9) How many distinct 6-digit integers are there such that each integer contains exactly two 0s? (for example 202204, 31001; also the leading digit should not be 0).

Solution: There are C(5, 2)=10 ways to place two 0s in 5 positions. The remaining 4 positions can be any nonzero digits. Therefore, there are $10 \cdot 9^4 = 65610$ distinct such 6-integers.

(10) Adam is in class A of 20 students, and Ben is in class B of 30 students. The average of grades for a course for class A is 1 point higher than that of class B. If Adam and Ben switch classes and carry their marks, the new averages of two classes are the same. What is the gap between Adam's mark and Ben's mark?

Solution: Suppose Adam's mark is x points higher than Ben's. After switching Adam's and Ben's grades, the two classes have the same average. This implies by switching back to the original classes, Class A gains x points while class B loses x points. Thus Class A gains $x/20$ points on average while Class B loses $x/30$. This difference is 1. That is, $x/20 + x/30 = 1$. Therefore $x = 12$.

(11) In a right triangle ABC, the lengths of 3 sides opposite to vertices A, B, C are a, b, c. If a perpendicular is drawn from the vertex of the right angle C to the AB and meets AB at D, find the length of CD in terms of a and b.

Solution: The right triangle CBD is similar to ABC. Thus $CD/BC = AC/AB$. But $AB = \sqrt{a^2+b^2}$. Therefore $CD = ab/\sqrt{a^2+b^2}$. We can also use the area of triangle ABC: $\frac{1}{2}ab = \frac{1}{2}CD \cdot AB$ to get the same answer.

(12) A sequence $\{a_n\}$ satisfies is $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. If $a_1 = 3$ and $a_6 = 29$, then what is a_7?

Solution: Suppose $a_2 = a$. Then $a_3 = a+3, a_4 = a_3+a_2 = 2a+3, a_5 = a_4+a_3 = 3a+6, a_6 = a_5 + a_4 = 5a+9$. $5a+9 = 29$, $a = 4$. Therefore $a_7 = a_6 + a_5 = 29 + (3 \cdot 4 + 6) = 47$.