Solutions to Junior Contest

Multiple Choice Problems

1. D
2. B
3. C
4. E
5. B
6. B
7. C
8. B
9. E
10. A
11. A
12. A
13. C
14. D
15. E

Full Solution Problems

1. Two natural numbers a, b have least common multiple being 1260. Find the minimum possible value of $a+b$.

   Solution If $\gcd(a,b) = d > 1$, then $a = dx, b = dy$ for some natural numbers x, y. We see that $\text{lcm}(a,b)$=$\text{lcm}(x,dy)$, but $a+b > x+dy$ and $\gcd(x,dy) = 1$. This means with the same lcm, we can get smaller sum if gcd of two numbers is 1. So we can just assume $\gcd(a,b) = 1$ and hence $ab = 1260$. We know $(\sqrt{a}-\sqrt{b})^2 = a+b-2\sqrt{ab} = a+b-2\sqrt{1260}$. To minimize $a+b$, we just need to minimize $(\sqrt{a}-\sqrt{b})^2$, that is, to minimize $\sqrt{a}-\sqrt{b}$ for $a > b$. $1260 = 36 \cdot 35$. Let a, b, u, v be integers such that $ab = uv = 1260$. If $a > u > 35.5$, then $b = 1260/a < 1260/u = v < 35.5$. This means $\sqrt{a}-\sqrt{b} > \sqrt{u}-\sqrt{v}$. This means we will get a smaller sum $a+b$ if a, b are closer to 35.5. Therefore the smallest sum is $36+35 = 71$.

2. Alice takes 1 step forward, 2 backwards, 3 forwards, 4 backwards, and so on. Each step is 60 cm long. How many steps are needed to reach 60 meters away from the starting point?

   Solution Alice takes alternating steps forwards and backwards. Every two moves, Alice's position is 1 step backwards. Let k be a positive integer. Her position after taking $2k-1$ steps forwards is $1-2+3-4+\cdots-(2k-2)+2k = -(k-1)+2k = k+1$ steps forwards. If she is 60 meters away from the starting point, her position needs to be 100 steps forwards. Thus $k+1 = 100$, and $k = 99$. The total steps she takes to reach this position is the sum of all steps she takes forwards and backwards, that is, $1+2+3+4+\cdots+(2k-2)+(2k-1) = (2k)(2k-1)/2 = k(2k-1) = 100 \cdot 199 = 19900$.

3. Adam and Ben play the following game. There is initially 2024 stones, and Adam and Ben alternate in removing some stones. Adam goes first, and chooses a positive divisor of 2024 and removes that many stones from the pile. Then Ben chooses a positive divisor of the number of stones remaining, and removes that many. They continue in this manner, and the player who takes the last stone loses. Show that there is a winning strategy for one of Adam and Ben, and describe the strategy, making clear that it wins.

   Solution Adam has a winning strategy. Adam starts with an even number. He takes 1 stone for the first move. Ben has an odd number. All the positive divisors of an odd number are odd numbers. He can take all stones, which results in his lose. Otherwise, he has to remove an odd number of stones, which means Adam will get always an even number. Each time, Adam always takes 1 stone to force Ben lose the game or leave an even number to Adam. The even number gets smaller each time. When there are two stones, Adam takes 1 stone, leave the last stone to Ben. After finite moves, Adam wins the game.