Solutions to Senior Contest

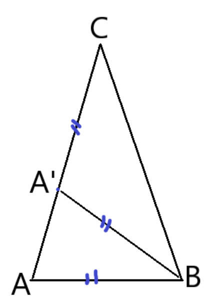
Multiple Choice Problems

1. D
2. B
3. D
4. E
5. B
6. D
7. D
8. A
9. C
10. E
11. B
12. A
13. A
14. D
15. B

Full Solution Problems

1. In the isosceles triangle $\triangle ABC$, we have $AC = BC > AB$. Let A' be the point on AC such that $AB = A'C = BA'$. How large is the angle $\angle ACB$?

   Solution Set $\angle ACB = x$. Since $A'C = A'B$, $\angle CBA' = x$. We know $\angle AA'B = \angle ACB + \angle CBA' = 2x$. Since $A'B = AB$, we have $\angle A'AB = \angle AA'B = 2x$. In the isosceles triangle $\triangle ABC$, $\angle CBA = \angle CAB = 2x$. The sum of three angles in a triangle is $180°$. So we have $x + 2x + 2x = 180°$ and $\angle ABA' = x = 36°$.

2. For how many values of k does the polynomial $p(x) = x^2 - kx + 36$ have at least one positive integer root less than 1000?

   Solution Let x be a positive integer less than 1000.

   x is a root of $p(x) \Leftrightarrow x^2 - kx + 36 = 0 \Leftrightarrow k = \frac{x^2+36}{x}$.

For $x = 1, 2, \cdots, 999$, all $p(x)$ with $k = \frac{x^2+36}{x}$ will have a positive integer root x less than 1000. For different x, we may have the same value k. Let x, y be distinct positive integers less than 1000. If

$$\frac{x^2 + 36}{x} = \frac{y^2 + 36}{y},$$

then $x^2y + 36y = xy^2 + 36x$, $(x - y)(xy - 36) = 0$. As $x \neq y$, we have $xy = 36$. There are 4 pairs of positive integer solutions to this equation: $\{1, 36\}, \{2, 18\}, \{3, 12\}, \{4, 9\}$. These are the only pairs to make k the same value. Therefore there are $999 - 4 = 995$ distinct values of k so that $p(x)$ has at least one positive integer root less than 1000.

3. On Anthrax island, there are 42 red chameleons and 49 blue chameleons and 59 green chameleons. Whenever two chameleons of different colour meet, they will both immediately change to the third colour. Is it possible that all of the chameleons on the island will ever be of the same colour?

   Solution Notice that when divided by three, the numbers 42, 49, and 59 leave remainders 0, 1, and 2 respectively. Let r, g, and b be the numbers of red, green, and blue chameleons respectively. If a pair of chameleons meet, two of these numbers are decreased by 1, and the other is increased by 2. If a number has remainder 0 when divided by three, adding 2 will give a number that will leave a remainder of 2, while subtracting 1 will also give a number that would leave a remainder of 2. If a number has remainder 1 when divided by three, adding 2 will give a number that will leave a reminder of 0, while subtracting 1 will also give a number that will leave a remainder of 0. If a number has remainder 2 when divided by three, adding 2 will give a number that will leave a reminder of 1, while subtracting 1 will also give a number that will leave a remainder of 1. Thus whenever two chameleons meet, if the original numbers had remainders 0, 1, and 2, the new numbers will also have remainders 0, 1, and 2, in some other order. Thus after any combination of chameleon meetings, the numbers r, b, and g will always have distinct remainders after dividing by three. It follows that r, b, and g are always distinct numbers, and we can never have two of them zero. Thus the chameleons will never all be the same colour.