An Investigation of Fractals and Fractal Dimension

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1 Introduction

1.1 Fractals in Nature

There are many fractals in nature. Most of these have some level of self-similarity, and are called self-similar objects. Basic examples of these include the surface of cauliflower, the pattern of a fern, or the edge of a snowflake. Benoit Mandelbrot (considered to be the father of fractals) thought to measure the coastline of Britain and concluded that using smaller units of measurement led to larger perimeter measurements. Mathematically, as the unit of measurement decreases in length and becomes more precise, the perimeter increases in length. For any unit of measurement $1/n$ we can use a smaller unit, namely $1/(n+1)$. Moreover, the perimeter is always getting larger; that is, it is always increasing. Does this always increasing sequence (measuring the coastline of Britain) converge or diverge?

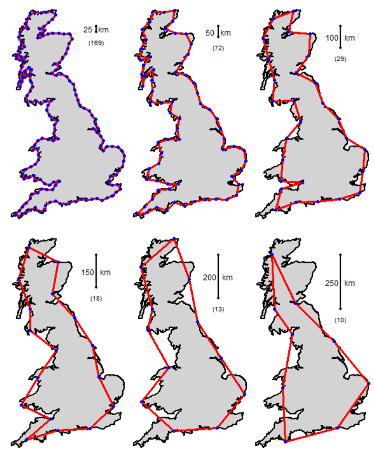


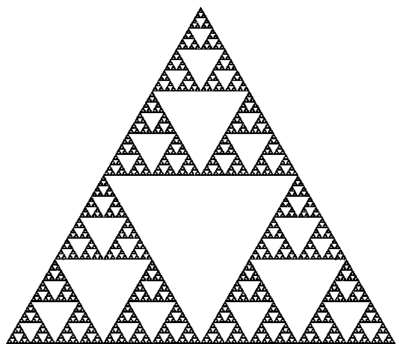
Figure 1: Coastline of Britain

This leads to the paradoxical question: Can a finite area be enclosed by an infinite length boundary? Can a finite volume be enclosed by an infinite surface area? Solutions to these questions can be found in the study of fractals.

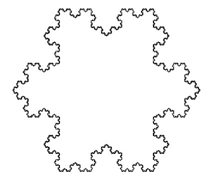
1.2 Mathematically Constructed Fractals

There are also many examples of fractals that are constructed mathematically. One of the most famous examples is the triadic Cantor Dust set. This set is created via the repeated deletion of the open middle third interval of a line segment. Another very popular example is the Seirpinski Triangle (Figure 2). The von Koch snowflake is another famous fractal (Figure 2). Many fractals can be constructed through an iteration process and use self-similar shapes.

Take the von Koch snowflake as an example. Beginning with an equilateral triangle one deletes the middle third of each side segment. Then two lines, equal to the



(a) Sierpinski Triangle



(b) Von Koch Snowflake

Figure 2: Mathematical Fractals

length of the deleted segment, are placed to create an outward facing "triangle" on each of the original triangle sides. This process is repeated, thus forming "mountain peaks" built on "mountain peaks". Some fractals that are defined recursively are called dragons.

As we will see later on, there are also fractals in the complex plane. These are often created by inputting a complex number through a function and iterating the process. The most popular of these types of fractals are Julia sets. The crowning glory of fractals is the Mandelbrot set. The Mandelbrot set is the set of all $c \in \mathbb{C}$ where the function $f_c(z) = z^2 + c$, when iterated from $f_c(0)$, does not diverge.

1.3 Motivation and Method

The necessity of fractional dimensions arise when we try to quantize the "size" of certain sets that are not simple, and are often hard to conceptualize. For example: The Sierpinski Triangle (sometimes called the Sierpinski gasket to convey geometric qualities), which lies in a 2 dimensional plane, is created by tremas, and has zero area.

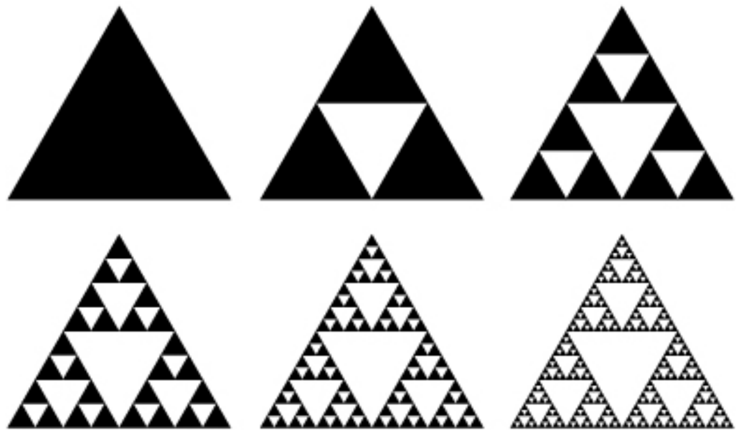


Figure 3: $S_k : k = 0, 1, ..., 5$

The set S_k has 3^k equilateral triangles in it, each with side length 2^{-k}. Hence, the total area is $3^k \times (2^{-k})^2 \times \sqrt{3}/4$. As $k \to \infty$ the total area converges to 0. Yet, the line segments that form the borders of each triangle are never deleted. Therefore, the set contains all those line segments. Again, the set S_k has 3^k triangles, each with 3 sides of length 2^{-k} meaning that the total perimeter of S_k is $3^k \times 3 \times 2^{-k}$. As $k \to \infty$ the total perimeter length diverges to ∞. Hence, the Sierpinski Triangle is a set which simultaneously: occupies 2 dimensional Euclidean space, has zero area and is not 2 dimensional (is less than 2), and has a perimeter (boundary) of infinite length. In a sense, one could imagine a disk in $\mathbb{R}^2$ being created by an infinitely long spiral, thus implying that the filled in disk has infinite length and is actually 2-dimensional. Therefore, it seems that the dimension of the Sierpinski Triangle is greater than 1 and less than 2. In 1919 Hausdorff proposed fractional dimensions which would accurately describe the properties of some of these sets that border on geometrical chaos [1]. These fractional dimensions would soon lead to a world of mathematics lying between topology and measure theory, a world that seeks to quantize roughness in such a way so as to manipulate and utilize roughness with precision.

Thus far, we have discussed fractals in a very imprecise fashion. Fractals are much more than the select few aesthetically appealing objects discussed. In order to utilize fractals and understand them mathematically we will need a rigorous approach with clear, precise definitions. To do this we must first become acquainted with some basic topology and measure theory. Through this study we will come to understand dimension. More specifically, we shall investigate sets that have non-integer dimension.

2 Basic Topology

There are a few powerful generalizations that can be made to help understand certain sets. The investigation of fractals lies almost entirely within metric spaces which are compact and separable.

2.1 Metric Spaces

We define a **Metric space** in the usual way. Let S be a set, paired with a function $\rho : S \times S \to [0, \infty)$. If $\rho(x, y)$ satisfies,

$$\begin{aligned}
&\rho(x, y) = 0 \Leftrightarrow x = y;\\
&\rho(x, y) = \rho(y, x);\\
&\rho(x, z) \leq \rho(x, y) + \rho(y, z);
\end{aligned}$$

then S is a Metric space.

A few other helpful definitions that will be used later are the diameter of a set and the distance between two sets.

Definition 2.1. *The* **diameter** *of a subset A of a metric space S is*

$$diam\ A = \sup\{\rho(x, y) : x, y \in A\}.$$

In other words, the diameter of A is the distance between the two most distant points of A, if such points exist. The diameter of a set will be a crucial form of measurement used later to compute the Hausdorff dimension. The **convex hull** of any set A has the same diameter as A itself. It should be noted here that the diameter of an open set A is equivalent to the diameter of the closure of the set, $\overline{A}$.

Definition 2.2. *If A and B are nonempty sets in a metric space S, we define the* **distance** *between them by*

$$dist(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}.$$

The distance between two sets is the distance between the two closest possible points, one in A and one in B, if such points exist.

It can be seen that if $A \cap B \neq \emptyset$, then $dist(A, B) = 0$ because they share at least one common element. One should also note however, that $dist(A, B) = 0$ does not imply $A \cap B \neq \emptyset$. That is, sets may have empty intersection, but still have zero distance, despite not sharing any elements. For example, let $A = (0, 1)$ and $B = [1, 2]$. Then $dist(A, B) = 0$ and $A \cap B = \emptyset$.

The **closure** of a set A is the set $\overline{A}$, comprised of A and all the accumulation points of A. A set $A \subseteq B$ may be called "**dense** in B" iff it's closure is equivalent to B, that is, $\overline{A} = B$. **Boundary points** of A are those points that lie in $\overline{A}$ as well as $\overline{S \backslash A}$. The boundary of A is the set of all boundary points, denoted ∂A.

We define the **open ball** centred at x with radius $r > 0$ as $B_r(x) = \{y \in S : \rho(x, y) < r\}$ where ρ is a metric in the metric space.

When discussing topology one must consider homeomorphisms. One such collection of homeomorphisms that are of particular interest are similarities. A function $h : S \to T$ is a **similarity** iff there is a positive number r such that

$$\rho(h(x), h(y)) = r\rho(x, y)$$

for all $x, y \in S$. The number r is the **ratio** of h. Two metric spaces are **similar** iff there is a similarity of one onto the other. It is important to note that similarities are continuous functions. In n-dimensional Euclidean space we have a wide selection of similarities: translations, reflections, rotations, dilations, and translation-reflections. We will revisit ratios and similarities in later sections.

From the Heine-Borel theorem we know that $[a, b] \in \mathbb{R}$ is a compact set. We also know that closed and bounded sets in $\mathbb{R}^d$ are compact. For our study of fractals, we will limit our investigation to compact sets.

Definition 2.3. *A metric space S is called* **sequentially compact** *iff every sequence in S has at least one cluster point in S.*

Definition 2.4. *A metric space S is called* **complete** *iff every Cauchy sequence in S converges in S.*

Definition 2.5. *Furthermore, suppose S and T are metric spaces. Let $f_n : S \to T$ be a sequence of functions, and let $f : S \to T$. The sequence f_n* **converges uniformly**

on S to the function f iff for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall x \in S$ and $\forall n \geq N$ we have $\rho(f_n(x), f(x)) < \epsilon$.

2.2 String Spaces

Many fractals have topological properties that are preserved by homeomorphisms. In particular, we may create homeomorphisms between a fractal and a sequence of letters, called strings. We let the space of strings, denoted $E^{(\omega)}$, be the set of infinite strings consisting of letters from the set $E = \{\alpha, \beta, \gamma ...\}$. The set $E^{(n)}$ will denote the set of strings with length n. We further assign a useful metric $\rho_{1/2}$ to the set $E^{(\omega)}$ to create a metric space. This will allow us to compare "closeness" of these elements.

Example: Let $E = \{0, 1\}$ and $\sigma, \tau \in E^{(\omega)}$. If $\sigma = \tau$, then $\rho_{1/2}(\sigma, \tau) = 0$. If $\sigma \neq \tau$, then there is a first letter they disagree on. There is a string, α, that consists of the longest common prefix such that $\sigma = \alpha\sigma'$ and $\tau = \alpha\tau'$. Let k be the length of that longest common prefix, α. We then define

$$\rho_{1/2}(\sigma, \tau) = (1/2)^k$$

The three properties of a metric space can be easily verified.

Remark: The space $E^{(\omega)}$ of infinite strings from the alphabet {0,1} is compact and complete under the metric $\rho_{1/2}(\sigma, \tau) = (1/2)^k$.

Example: One can consider the Cantor set as $E^{(\omega)}$ using an addressing function, $h : E^{(\omega)} \to C$. Here we have that $E = \{0, 1\}$ and the Cantor set in $\mathbb{R}$ is written in base 3 using 0's and 2's. Let $\sigma = .011111\cdots$ and $\tau = .011000\cdots$ be infinite strings in $E^{(\omega)}$. Then $h(\sigma) = (0.022222\cdots)_3 = 9/27 \in C$ and $h(\tau) = (0.022000\cdots)_3 = 8/27 \in C$. In $C \subset \mathbb{R}$ we understand that the points $9/27$ and $8/27$ are separated by a distance of $1/27$. However, in $E^{(\omega)}$, we may use the metric ρ to calculate the distance between σ and τ. That is,

$$\rho_{1/2}(\sigma, \tau) = (1/2)^3 = 1/8$$

since σ and τ have a common prefix of length $k = 3$.

Remark: The infinite string space can be intuitively understood to be a tree (a binary tree if E has two elements, a ternary tree if E has three elements, etc). Often, string spaces can help us to grasp qualities about a fractal that are not easily seen otherwise. The tree helps us visualize at what level do the points brach away from one another.

Example: One could also let $E = \{L, U, R\}$ and define an addressing function $h : E^{(n)} \to S_n$, where S_n is the Sierpinski triangle with 3^n triangles. The three alphabetical letters: L, U, and R correspond with the left, upper and right triangles of the Sierpinski triangle.

2.3 Hausdorff Metric

The Hausdorff Metric is quite powerful as it is a definition of a metric that applies to sets, rather than points.

Let S be a metric space. Let A and B be subsets of S. We say that A and B are within **Hausdorff distance r** of each other iff every point of A is within distance r of some point of B, and every point of B is within distance r of some point of A.

This idea can be made into a metric, called the **Hausdorff metric**, D. If A is a set and $r > 0$, then the **open r-neighborhood*** of A is $N_r(A) = \{y \in S : \rho(x, y) < r \text{ for some } x \in A\}$.

Definition 2.6. *Given two sets $A, B \subseteq S$ we define the* **Hausdorff metric** D : $D(A, B) = \inf\{r > 0 : A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\}$. *By convention,* $\inf \emptyset = \infty$.

Suppose that S is the Sierpinski triangle, and S_k is the Sierpinski triangle with 3^k triangles in it. The sequence S_k actually converges to S in the sense that the Hausdorff metric tends to 0.

Due to the fact that the Hausdorff metric does not define a metric with the three classic requirements we will restrict our applications of D to nonempty compact sets where indeed the Hausdorff metric is a metric. The Hausdorff metric is a metric defined on sets, not points.

*To be used to calculate covering dimension

3 Topological Dimension

There are numerous elementary sets which are easy to categorize into integer dimensions. Points are zero dimensional, curves are one dimensional, surfaces are two dimensional and solids are three dimensional. But we run into problems when we consider sets with more complicated topology, composed of combinations of points, curves, surfaces, and solids. In this section, we will define one helpful dimension called the covering dimension, which will be useful in many situations. We continue with some definitions in topology.

3.1 Refinement

Definition 3.1. *If* $\mathcal{A}$ and $\mathcal{B}$ *are two collections of sets, we say that* $\mathcal{B}$ *is* **subordinate** *to* $\mathcal{A} \Leftrightarrow \forall B \in \mathcal{B}\ \exists A \in \mathcal{A}$ *with* $B \subseteq A$*. Let* S *be a metric space, and let* $\mathcal{A}$ *be an open cover of* S*. A* **refinement** *of* $\mathcal{A}$ *is an open cover* $\mathcal{B}$ *of* S *that is subordinate to* $\mathcal{A}$*. We also say say* $\mathcal{B}$ **refines** $\mathcal{A}$*.*

When we later compute the Hausdorff dimension, we will use refinements of a family of sets such that we can have open covers $\mathcal{B}$ of S that are subordinate to open covers $\mathcal{A}$ of S.

3.2 Zero-dimensional Spaces

Let S be a metric space. A set $A \subseteq S$ is called a **clopen** set iff it is both open and closed. A **clopen partition** of S is an open cover of S consisting of disjoint clopen sets.

Theorem 3.2. *The metric space* S *is* **zero-dimensional** *iff every finite open cover of* S *has a finite refinement that is a clopen partition.*

If S is finite, then S is zero-dimensional because we may form a finite refinement of S that is a clopen partition by using each of the singletons, $\{x\}$, which on their own are clopen sets that partition S.

Theorem 3.3. *Let* S *be a compact metric space.* S *is zero-dimensional iff there is a base for the topology of* S *consisting of clopen sets.*

The only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$ itself. Hence, we do not have a finite refinement that is also a clopen partition. Therefore, $\mathbb{R}$ is not zero-dimensional.

Now suppose there are two zero-dimensional sets. What is the dimension of their union? If $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are zero-dimensional, then can we logically conclude that $\mathbb{R} \equiv (\mathbb{Q}) \cup (\mathbb{R}\backslash\mathbb{Q})$ is one-dimensional as we might guess? But, the union of $U = \{1\}$ and $V = \{2\}$, both zero-dimensional, is $U \cup V = \{1, 2\}$ which has a finite refinement that is a clopen partition showing that $U \cup V$ is also zero-dimensional. There is a logical sum theory when we restrict it to the union of closed sets.

Sum Theorem

Theorem 3.4. *Let S be a metric space. Let U and V be closed sets in S. If U and V are both zero-dimensional, then $U \cup V$ is zero-dimensional.*

3.3 Covering Dimension

Let $\mathcal{A}$ be a family of sets and let $n \geq -1$ be an integer. We define the **order** of $\mathcal{A}$ to be the smallest n such that the intersection of any $n+2$ sets from $\mathcal{A}$ have empty intersection.

As seen in Figure 4, one may cover the curve using disks in such a way such that any three disks will have empty intersection. However, one could select two disks that have non-empty intersection. The order of the open cover is 1 since any 3 of the sets have empty intersection.

We say that S has **covering dimension** $\leq n \Leftrightarrow$ every finite open cover of S has a refinement with order $\leq n$. The covering dimension is $n \Leftrightarrow$ the covering dimension is $\leq n$ but not $\leq n-1$. We denote the covering dimension of S as $Cov\ S = n$ or $Cov\ S = \infty$. Some literature calls this Lebesgue dimension, but due to the intuitive meaning of the word "covering" we will use covering dimension.

Figure 4 has covering dimension 1, since the finite open cover (the set of disks) has a refinement with order ≤ 1 but not ≤ 0.

Another concept that will be of help in forming refinement covers of a particular size will be the cover's mesh. Let $\mathcal{A}$ be a cover. We define the **mesh** of $\mathcal{A}$ to be $sup_{A \in \mathcal{A}}$ diam A.

In subsection 3.2 we showed that finite sets are zero-dimensional. We also showed that some infinite sets are also zero-dimensional. It can be shown that for a non-

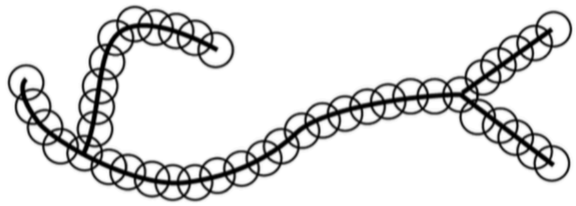


Figure 4: Disks covering a line

empty set S, $Cov\ S = 0 \Leftrightarrow S$ is zero-dimensional as defined above. This shows that covering dimension and the definition for zero-dimensional sets agree with one another.

Sum Theorem: Earlier we had that the union of two zero-dimensional closed sets was a zero-dimensional set. We can extend the sum theorem to 1-dimensional closed sets.

Theorem 3.5. *Let S be a metric space, and let U, V be closed subsets. If $Cov\ U \leq 1$ and $Cov\ V \leq 1$, then $Cov\ (U \cup V) \leq 1$.*

Theorem 3.6. *Let S be a compact metric space and let $n \geq 1$ be an integer. Then $Cov\ S \leq n$ if and only if for every $\epsilon > 0$, there is an open cover of S with order $\leq n$ and mesh $\leq \epsilon$.*

Theorem 3.7. *If $T \subseteq S$, where S is a metric space, then $Cov\ T \leq Cov\ S$.*

In subsection 3.2, we showed that $\mathbb{R}$ is not zero-dimensional and hence $Cov\ \mathbb{R} \geq 1$. We can also show $Cov\ \mathbb{R} \leq 1$, implying that $Cov\ \mathbb{R} = 1$. We work with the closed interval $[0, 1]$ in $\mathbb{R}$ for now. Let $\epsilon > 0$ be given and let $n \in \mathbb{N}$ such that $(1/n) \leq (\epsilon/2)$. Then we can have an open cover of $[0, 1]$ by

$$\{((k-1)/n, (k+1)/n) : k \in \mathbb{Z}\}$$

with mesh $\leq \epsilon$ and order 1. Though we examined $[0, 1]$ we simply extend this to the whole real line. This shows that $Cov\ \mathbb{R} \leq 1$. Therefore, $Cov\ \mathbb{R} = 1$.

Example: The Sierpinski triangle, S, has covering dimension 1. We know that S contains lines so $Cov\ S \geq 1$. Now we must show $Cov\ S \leq 1$. To do this we use Theorem 3.6 and show that for every $\epsilon > 0$, there is an open cover of S_k with order $\leq n$ and mesh $\leq \epsilon$. Let S_k be the Sierpinski triangle with 3^k filled in triangles, each with side length 2^{-k}. For any given k, any two triangles share a corner point or are within $2^{-k}/(\sqrt{3}/2)$ of each other. We can make an open cover of S_k using r-neighbourhoods of the triangles in S_k, where we let $0 < r < 2^{-k}/\sqrt{3}$. The open cover comprised of r-neighbourhoods of each of the triangles intersect with each other. However, any $3 = (1+2)$ r-neighbourhoods have empty intersection. Yet, we can select two r-neighbourhoods that have non-empty intersection. This shows us that we have a cover of S_k with order 1. Furthermore, the mesh is the supremum diameter of all the sets in the open cover. Since they are all r-neighbourhoods of triangles with side length 2^{-k} we have that the mesh of each r-neighbourhood is $2^{-k} + 2r < 2^{-k} + 2(2^{-k}/\sqrt{3})$. Therefore, given $\epsilon > 0$ we set k large enough such that $2^{-k}+2(2^{-k}/\sqrt{3}) < \epsilon$ and make the open cover of the set S_k with the r-neighbourhoods as defined above. Using Theorem 3.7 one can see that if $S \subseteq S_k$, where S is the Sierpinski triangle, then $Cov\ S \leq Cov\ S_k$. Since $Cov\ S_k = 1$ and $Cov\ S \geq 1$ we must have $Cov\ S = 1$.

4 Measure Theory

4.1 Outer/Inner Lebesgue Measure

For intervals in $\mathbb{R}$ of the form (a, b), $(a, b]$, $[a, b)$, $[a, b]$, their **length** is $b-a$ whenever $a < b$.

Definition 4.1. *Let $A \subseteq \mathbb{R}$. The* **Lebesgue outer measure** *of A is obtained by covering A with countably many half-open intervals of total length as small as possible. In symbols, $\overline{\mathcal{L}}(A) = \inf \sum_{j=1}^{\infty}(b_j - a_j)$ where the infimum is over all countable families $\{[a_j, b_j) : j \in \mathbb{N}\}$ of half-open intervals with $A \subseteq \cup_{j\in\mathbb{N}}[a_j, b_j)$.*

For any interval A, $\overline{\mathcal{L}}(A)$ is the length of A. This can be proved for open, closed, half-open, and half-closed intervals. The Lebesgue outer measure was achieved from the outside of a set inward, covering a set in $\mathbb{R}$ with half open intervals. Similarly, the Lebesgue inner measure will be achieved from the inside of the set outward.

Definition 4.2. *Let $A \subseteq \mathbb{R}$. The* **Lebesgue inner measure** *of a set A is $\underline{\mathcal{L}}(A) = \sup\{\overline{\mathcal{L}}(K) : K \subseteq A, K \text{ is compact}\}$*

It can be shown that for any $A \subseteq \mathbb{R}$ we have $\underline{\mathcal{L}}(A) \leq \overline{\mathcal{L}}(A)$. We say that a set A is **Lebesgue measurable** when $\underline{\mathcal{L}}(A) = \overline{\mathcal{L}}(A)$. This equation, however, is not generally true. When it is, we denote the Lebesgue measure of A as $\mathcal{L}(A)$.

The union of finitely many disjoint Lebesgue measurable sets is itself measurable and its length is equal to the summation of each of the finitely many Lebesgue measures.

It should be noted that compact subsets, closed subsets, and open subsets of $\mathbb{R}$ are Lebesgue measurable. Some simple algebraic properties will help us down the road. (1) Both $\emptyset$ and $\mathbb{R}$ are Lebesgue measurable. (2) If A and B are Lebesgue measurable, then $\mathbb{R}\backslash A$, $A \cup B$, $A \cap B$, and $A \backslash B$ are all Lebesgue measurable. (3) If A_n is Lebesuge measurable for $n \in \mathbb{N}$, then $\cup_{n\in\mathbb{N}} A_n$ and $\cap_{n\in\mathbb{N}} A_n$ are Lebesgue measurable (this holds because these are countable unions and countable intersections).

Remark If $A \subseteq B$, then $\overline{\mathcal{L}}(A) \leq \overline{\mathcal{L}}(B)$

Example: We can calculate the Lebesgue measure of the Cantor set. Let C be the Cantor set and C_n be the set with 2^n segments, each of length $(1/3)^n$. Note that $C \subseteq C_n$. We can see that as $n \to \infty$ we have $\mathcal{L}(C_n) = 2_n \cdot 3^{-n} \to 0$, thus sandwiching $0 \leq \mathcal{L}(C) \leq 0$.

4.2 Carathéodory Measurability

Definition 4.3. *A set $A \subseteq \mathbb{R}$ is* **Carathéodory measurable** *iff*

$$\overline{\mathcal{L}}(E) = \overline{\mathcal{L}}(E \cap A) + \overline{\mathcal{L}}(E \setminus A)$$

for all sets $E \subseteq \mathbb{R}$.

For sets in $\mathbb{R}$, Carathéodory measurablity and Lebesgue measurability are equivalent.

Theorem 4.4. *Let $A \subseteq \mathbb{R}$ be Lebesgue measurable, and let a similarity $f : \mathbb{R} \to \mathbb{R}$ with ratio r be given. Then $f[A]$ is Lebesgue measurable and $\mathcal{L}(f[A]) = r\mathcal{L}(A)^*$.*

*Helpful in Section 5

4.3 Two dimensional Lebesgue Measure

In $\mathbb{R}$ we used half-open intervals to measure the length of sets. In $\mathbb{R}^2$ we will use rectangles. We define a rectangle R as

$$R = [a, b) \times [c, d) = \{(x, y) \in \mathbb{R}^2 : a \leq x < b, c \leq y < d\}$$

for $a \leq b$ and $c \leq d$. The area of this rectangle is $\mathbf{c}(R) = (b - a)(d - c)$.

A rectangle of the form $R = [a, b) \times [c, d)$, where $a < b$ *and* $c < d$, is Lebesgue measurable and $\mathcal{L}^2(R) = (b - a)(d - c)$.

For sets in $\mathbb{R}^2$ that are not rectangular we use the usual approximating scheme of little rectangles within the set. The two-dimensional Lebesgue measure will be equal to the area.

Furthermore, we may have a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is a similarity with a ratio r. Suppose $A \subseteq \mathbb{R}^2$ is Lebesgue measurable, then so is $f[A]$, and $\mathcal{L}^2(f[A]) = r^2\mathcal{L}^2(A)$.

Extending the Lebesgue measure to d-dimensional Euclidean space can be carried out by using **hyper-rectangles**, $R = [a_1, b_1) \times \cdots \times [a_d, b_d)$, and thus calculating a "hyper volume", $\mathbf{c}(R) = \prod_{j=1}^{d}(b_j - a_j)$.

5 Self-similarity

Many fractals have a level of self-similarity in their geometric composition. Segments of the boundary may be composed of smaller scale segments of itself. The Sierpinski gasket can be seen to be self-similar in that the large triangle is the same as one of the three triangles composing the whole. In this section we will define our first definition of dimension that will be used as a fractal dimension. We also define an iterated function system which will help us efficiently specify many sets that we are interested in.

5.1 Ratios

A **ratio list** is a finite list of positive numbers $(r_1, r_2, \cdots, r_n)$. An **iterated function system** realizing a ratio list $(r_1, r_2, \cdots, r_n)$ in a metric space S is a list $(f_1, f_2, \cdots, f_n)$, where $f_i : S \to S$ is a similarity with ratio r_i.

A nonempty compact set $K \subseteq S$ is an **invariant set** for the iterated function system $(f_1, f_2, \cdots, f_n)$ iff $K = f_1[K] \cup f_2[K] \cup \cdots \cup f_n[K]$. Examples of an invariant set include the Cantor set and the Sierpinski gasket. We may also say that K is a set attractor*.

Definition 5.1. *The* **sim-value** *of a ratio list* $(r_1, r_2, \cdots, r_n)$ *is the positive number* s *such that* $r_1^s + r_2^s + \cdots + r_n^s = 1$.

If $r_i < 1$ for all i, then the ratio list is **contracting** or **hyperbolic**. The **similarity dimension**, s, of a set K satisfies a self-referential equation of the type $K = \cup_{i=1}^n f_i[K]$ where there is a hyperbolic iterated function system of similarities whose ratio list has sim-value s. We must be careful here because the similarity dimension is defined for a set AND an iterated function systems realizing a ratio list for that particular set. That is to say that the sim-value of a particular set is not unique†.

5.2 Examples and Calculations

Consider the interval $[a, b]$. The interval can be understood to be the union of two smaller intervals, $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Each of these smaller intervals are similar to the whole interval. Moreover, they have a ratio $r = 1/2$. Therefore, the ratio list is $(1/2, 1/2)$. The sim-value is the solution to the equation

$$(1/2)^s + (1/2)^s = 1$$

$$2(1/2)^s = 1$$

so $s = 1$. Therefore, the similarity dimension is 1. This makes sense because one naturally expects any interval on $\mathbb{R}$ to have dimension 1.

However, the interval $[a, b]$ may also be written as $[a, (a+2b)/3] \cup [(2a+b)/3, b]$, which corresponds to a ratio list $(2/3, 2/3)$ and yields a similarity dimension larger than 1. This reveals some of the limitations of similarity dimension. One of the limitations is that the similarity dimension is actually based on the iterated function system rather than the set itself, which causes problems when there are multiple different

*The chaos game, described in section 5.3, played with 3 points, will eventually give a beautiful picture of the Sierpinski gasket, illustrating that the Sierpinski gasket is an attractor because it attracts these points to itself.

†See the Sierpinski example below

iterated function systems to define a set, as seen above. Hence, a useful definition for dimension will limit overlapping elements as desired.

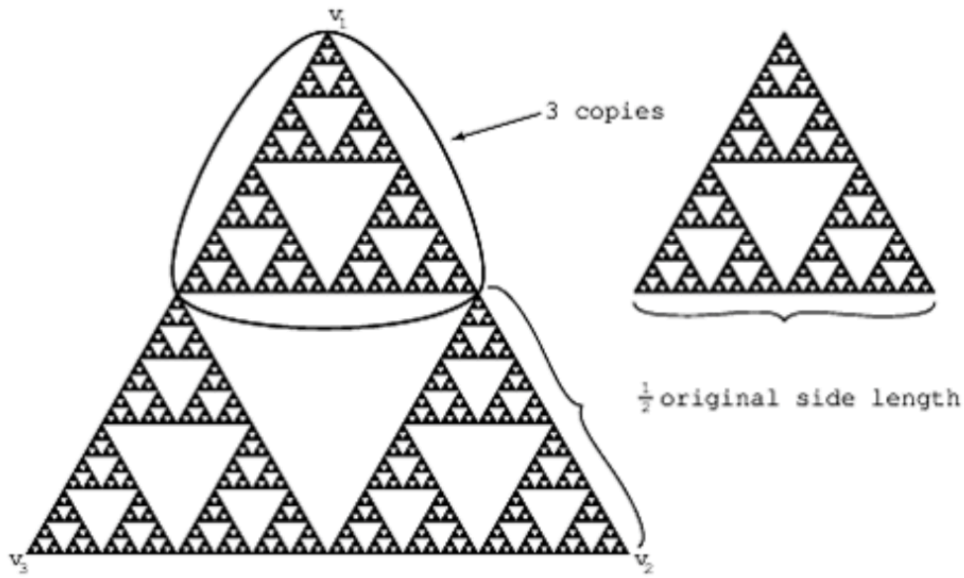


Figure 5: Sierpinski Similarity Ratio

Sierpinski gasket: Upon examination one can see that the large triangle is composed of three smaller triangles (left, upper, right), each of which are scaled down to half the size (half the side length). See Figure 5. One can easily compute the similarity dimension of the Sierpinski triangle and the corresponding natural iterated function system. The ratio list is $(1/2, 1/2, 1/2)$ because there are three copies (f_L, f_U, f_R), each scaled by $1/2$. The sim-value is the solution to the equation

$$(1/2)^s + (1/2)^s + (1/2)^s = 1$$

$$3(1/2)^s = 1$$

so $s = (log\ 3)/(log\ 2) \approx 1.585$. Therefore, the similarity dimension is approximately 1.585.

As cautioned above, one could have a ratio list $(1/2, 1/2, 1/2, 1/2)$ and say that the four functions realizing those ratios are the left triangle, the upper triangle, the right triangle and the right triangle again. That is f_L, f_U, f_R, f_R. Thus giving us a sim-value of 2. This reveals that the similarity dimension on its own is not well defined.

Cantor dust set: The Cantor set is likewise composed of two smaller line segments, each of which are scaled down to half the length. Therefore, the ratio list is $(1/3, 1/3)$. The sim-value is $(1/3)^s + (1/3)^s = 1$, so $s = (log\ 2)/(log\ 3) \approx 0.631$.

Although this notion of fractional dimension is very quick and easy to compute, it has limiting restrictions. The set in question must have self similar attributes. Many fractals, notably the very rough ones, do not have self similar attributes. The sim-value is calculated for the set AND an iterated function system (which might have overlap). Therefore, the similarity dimension is not well defined for many sets, even simple sets such as a closed ball. When we are able to, we will compute fractional dimensions using similarity dimension, but it will prove unhelpful for many sets.

5.3 Chaos Game

It seems weird to have a subsection entitled "Chaos Game" within a section about self-similarity where you would think nothing chaotic happens. But we will see the connection quickly.

There is a fascinating way to program the creation of many fractals, which will reveal the intuitive meaning of "set attractor" very quickly. In a very general sense, one may choose n points in $\mathbb{R}^d$. Then, start the iteration process at any point $x \in \mathbb{R}^d$. From x, randomly choose (based on some probability) one of the n points and go a fraction of the distance towards it (i.e. 1/2, 1/3, etc). This then becomes the second point. Continue the random iteration over and over again, marking down each point as you go. Though it seems that one has chosen arbitrary points and are doing random movements, one will eventually create a fractal. That is, the iteration process will attract each new point x_i to lie in the fractal set, the set attractor.

As an example once again, the Sierpinski gasket can be created this way in just a few steps (or lines of code).
(1) Let v_1, v_2, and v_3 be the three corners of a triangle in $\mathbb{R}^2$.

(2) Choose any point $x_0 \in \mathbb{R}^2$.
(3) Move 1/2 the distance toward a randomly chosen vertex, v_1, v_2, or v_3.
(4) Let this new point be x_1.
(5) Repeat step (3) and (4) labeling each new point x_i where i is the iteration number.

Carrying on this process to infinity will produce the Sierpinski gasket. That is, using the sequence (x_i) created, the set $X_i = \{\cup_{i=1}^{n} x_i : x_i \in \mathbb{R}^2\} \to S$ as $n \to \infty$. Once again this convergence is in the sense of the Hausdorff metric where

$$\lim_{i \to \infty} D(X_i, S) = inf\{r > 0 : X_i \subseteq N_r(S) \text{ and } S \subseteq N_r(X_i)\} = 0$$

This is fascinating because a seemingly random process, first appearing chaotic, eventually creates such intricate detail and beautiful images; hence the name, “the chaos game”.

Altering the rules of the iterating function system can produce amazing computer generated graphics, including ferns (Barnsley leaf), trees, mountain landscapes, and many other objects. By incorporating certain probabilities one can generate simultaneously the randomness that nature has but also the incredible symmetry and orderliness that nature has, thus creating realistic graphics that are not all copies of the same pixelated image.

6 Fractal Dimension

Now that we have a little background in topology and measure theory we turn to fractal dimensions. We already have the similarity dimension at our disposal, though it is fairly weak and unhelpful. We also have the integer topological dimension, which we called the covering dimension. Therefore, we look to find more generalizable and precise definitions. We begin with the Hausdorff dimension and move to Packing dimension afterwards.

6.1 Hausdorff Measure

Let S be a metric space. Using the set function $\mathbf{c}_s(A) = (diam\ A)^s$ we define the **s-dimensional Hausdorff outer measure**, denoted $\overline{\mathcal{H}}^s$. As with Lebesgue measure, when we restrict this to the measurable sets it is called **s-dimensional Hausdorff measure**, denoted $\mathcal{H}^s$.

Recall that one may have a countable cover of a set $F \subseteq S$. Suppose $\mathcal{A}$ is a family of sets in a metric space S. $\mathcal{A}$ is a **countable cover** of a set F iff

$$F \subseteq \bigcup_{A \in \mathcal{A}} A,$$

and $\mathcal{A}$ is a countable family of sets. Furthermore, the cover $\mathcal{A}$ is an ϵ-cover for $\epsilon > 0$ iff diam $A \leq \epsilon$ for all $A \in \mathcal{A}$.

Definition 6.1. *We define the* **s-dimensional Hausdorff outer measure** *as*

$$\overline{\mathcal{H}}_\epsilon^s(F) = \inf \sum_{\mathrm{A} \in \mathcal{A}} (\mathrm{diam\ A})^{\mathrm{s}}, \tag{1}$$

where the infimum is over all countable ϵ-covers $\mathcal{A}$ of the set F.

In the 2 dimensional Euclidean space, the Hausdorff outer measure and Riemann sums of integration have many parallel properties.

Note that there are a huge amount of possible coverings. However, in calculating the Hausdorff outer measure a few things need to be true:
(1) All the sets $A \in \mathcal{A}$ must have a diameter $\leq \epsilon$,
(2) The family $\mathcal{A}$ must be countable, and of all the possible countable ϵ-covers we take the one that gives us the infimum when we sum up the diameters of the countably many sets.

Suppose that both $\mathcal{A}$ and $\mathcal{B}$ are two collections of set that form open covers of S, and that $\mathcal{B}$ is subordinate to $\mathcal{A}$. When computing the Hausdorff outer measure we always use the subordinate family of sets. (Recall definition 3.1 for the definition of refinement). We will have coverings of a set F that have mesh $\leq \epsilon$. Therefore, if a refinement of $\mathcal{A}$ exists, we use it to compute Hausdorff outer measure. This helps us get the smallest possible value when we sum up the diameters of the countably many sets.

If we carry out a limiting process with epsilon we obtain the s-dimesional Hausdorff outer measure. That is,

$$\overline{\mathcal{H}}^s(F) = \lim_{\epsilon \to 0} \overline{\mathcal{H}}_\epsilon^s(F) = \sup_{\epsilon > 0} \overline{\mathcal{H}}_\epsilon^s(F) \tag{2}$$

There are a few tips and shortcuts:
(i) To compute the Hausdorff measure for a compact set K we may use a finite cover.

(ii) If we replace a set A in a cover $\mathcal{A}$ of F by a subset of itself, $A' \subseteq A$, so that the result is still a cover, then $\sum\limits_{A \in \mathcal{A}} (\text{diam A})^{\text{s}}$ becomes smaller.
(iii) When it is convenient we may assume that the sets used in the covers $\mathcal{A}$ of the set F are subsets of F itself.

The Hausdorff measure essentially tries to efficiently cover a set F with a countable ϵ-cover such that we minimize needlessly covering elements not in F and that we minimize a "double" covering of elements in F.

Example: Let $F = \{0, 1\} \subset \mathbb{R}$. We can find a countable cover $\mathcal{A}$ of F, say $A_n = (n - \epsilon/2, n + \epsilon/2)$ where n is an integer. The smallest possible covering of F using sets from $\mathcal{A}$ are the two sets $A_1 \cup A_2 = (-\epsilon/2, \epsilon/2) \cup (1 - \epsilon/2, 1 + \epsilon/2)$. Using Lebesgue measure we see that the length of each set is $(n + \epsilon/2) - (n - \epsilon/2) = \epsilon$, that is $\mathcal{L}(A_n) = \epsilon$. Moreover, diam $A_n = \epsilon$. Therefore, by equations (1) and (2) as we take a limiting process with epsilon going to 0 we get that $\mathcal{H}^s(F) = 0$, where $F = \{0, 1\}$. Therefore, the Hausdorff dimension of the two element set $F = \{0, 1\}$ is 0. We will also show that $\mathcal{H}^0(F) = 2$, showing that the Hausdorff measure can be a cardinality measure.

The extension of the previous example is that for any finite set F we have $\mathcal{H}^s(F) = 0$ for all $s > 0$. This is because each finite point is covered by an open set from the family of sets $\mathcal{A}$ that we can make arbitrarily small. Then, when we sum up all the arbitrarily small diameters and take the infimum we can show that the set has Hausdorff outer dimension equal to 0.

We can also say something about 0-dimensional Hausdorff measure. We know that for any finite set F we have $\mathcal{H}^s(F) = 0$ for all $s > 0$. But if $s = 0$ we can see that Hausdorff dimension becomes a function describing the cardinality of the set. From our equations (1) and (2) we see that when $s = 0$ we have,

$$\mathcal{H}^0(F) = \begin{cases} n \text{ if } F \text{ has } n \text{ elements,} \\ \infty \text{ if F is infinite.} \end{cases}$$

This is a very nice property of the Hausdorff dimension. Now, we continue to investigate to see if it has more uses. Of particular interest is whether or not the Hausdorff dimension corresponds to Lebesgue measure in 1 dimensional Euclidean space.

Theorem 6.2. *The one dimensional Hausdorff measure $\mathcal{H}^1$ coincides with Lebesgue measure $\mathcal{L}$.*

This can be seen by setting $s = 1$ in the Hausdorff measure and comparing the formula to Lebesgue measure. The diameter of any set $A \in \mathcal{A}$ corresponds with the half-open intervals used in the Lebesgue measure.

What happens then when we examine the Hausdorff measure $\mathcal{H}^s(F)$ as a function of s for a given set F?

Theorem 6.3. *Let F be a Borel set. Let $0 < s < t$. If $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. If $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$.*

So the Hausdorff measure as a function of s decreases as s increases. Moreover, there is a unique critical value $s_0 \in [0, \infty)$ such that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } s < s_0; \\ 0 & \text{if } s > s_0. \end{cases}$$

As a function of s, the Hausdorff dimension of a set F is a step function. This unique critical value s_0 is what we will call the **Hausdorff dimension**, denoted $\mathbf{s_0 = \dim F}$. This is what we have been searching for; a precise definition of dimension that fits our elementary geometric definitions and can be abstracted to sets with non-integer dimension.

Conveniently, for $s = 0$ the Hausdorff dimension is a measure of cardinality. This is because the most efficient covering of a finite set is the collection of finitely many $\epsilon/2$-neighbourhoods of each of the elements. Raising $\epsilon/2$ to the power of 0 gives 1, so we essentially just count the number of elements to find the Hausdorff measure.

Let A, B be Borel sets. Some properties of Hausdorff dimension are:
(1) If $A \subseteq B$, then $\dim A \leq \dim B$.
(2) $\dim(A \cup B) = \max\{\dim A, \dim B\}$.

Remark 6.4. *As we had before, we may consider a function realizing a ratio. Let $f : S \to T$ be a similarity with ratio $r > 0$. Let s be a positive real number, and let $F \subseteq S$ where S is a metric space. Then $\overline{\mathcal{H}}^s(f[F]) = r^s \overline{\mathcal{H}}^s(F)$. So $\dim f[F] = \dim F$. Therefore, the Hausdorff dimension is not altered by similarity functions.*

6.2 Packing Measure

There is another way to define a set's dimension; packing dimension. Unlike the Hausdorff dimension, which sought to cover a set F in the most efficient way using open sets and minimizing overlap or excess covering, the packing dimension (which will be defined using the packing measure) works from the inside out. It seeks to put disjoint sets inside F in the most efficient way trying to maximize the space by filling (i.e. packing) as much of the set F as possible.

However, we must be careful how we pack. We could pack a square, F, using arbitrarily thin sets A_i. Though the square may not be that "big" we could evaluate $\sum_{i \in \mathbb{N}} (\mathrm{diam}\ \mathrm{A_i})^{\mathrm{s}}$ to be as large as we like and hence we would be misled. In Euclidean space the most natural packing shape is a cube, however we will use balls due to their general application to metric spaces. Furthermore, we may desire that the whole ball be contained in the set F we are measuring, but some sets contain no balls (i.e. a line in $\mathbb{R}^2$). Therefore, we only require that the centre of the ball be contained in the set F we are measuring.

Let S be a metric space and let us give some definitions that will help get to where we are going.

Definition 6.5. *A* **constituent** *in S is a pair (x, r), where $x \in S$ and $r > 0$. We may understand this to be the closed ball $B_r(x)$, where x is the center of the ball and is contained in S and r is the radius.*

Definition 6.6. *Let $E \subseteq S$. A* **packing** *of E is a countable collection $\prod$ of constituents, such that:*
(1) for all $(x, r) \in \prod$, we have $x \in E$;
(2) for all $(x, r), (y, s) \in \prod$ with $(x, r) \neq (y, s)$ we have $\rho(x, y) > r + s$.

That is, the center point of every constituent lies in the set E under investigation. Moreover, the second part ensures that two centers are far enough away from each other such that the two constituents don't overlap. The most common constituent is a ball and therefore a packing is a collection of disjoint sets of closed balls whose centers are in the set E.

Definition 6.7. *For $\delta > 0$, we say that a packing $\prod$ is* **δ-fine** *iff for all $(x, r) \in \prod$ we have $r \leq \delta$.*

Now, we have the definitions to work and form the **packing measure** definition.

Definition 6.8. *Let $F \subseteq S$, and let $\delta, s > 0$. Define the* **Packing Measure** *to be:*

$$\overline{\mathcal{P}}^s_\delta(F) = \sup \sum_{(x,r)\in\prod} (2r)^s$$

where the supremum is over all δ-fine packings $\prod$ of F.

As before, we take a limiting process, this time with $\delta \to 0$.

$$\overline{\mathcal{P}}^s_0(F) = \lim_{\delta\to 0} \overline{\mathcal{P}}^s_\delta(F) = \inf_{\delta>0} \overline{\mathcal{P}}^s_\delta(F) \tag{3}$$

We get a family $\overline{\mathcal{P}}^s_0$ of set functions indexed by s and once again there is a critical value for each set F under examination.

This unique critical value $s_0 \in [0, \infty)$ is such that, for a given set F we have:

$$\overline{\mathcal{P}}^s_0(F) = \begin{cases} \infty & \text{for all } s < s_0; \\ 0 & \text{for all } s > s_0. \end{cases}$$

We shall call this critical value s_0 the **packing index** for the set F.

By restricting our attention to measurable sets we obtain the **s-dimensional packing measure** $\mathcal{P}^s$. Moreover, the critical value, as above, is called the **packing dimension** of the set F. We denote the packing dimension as $\mathbf{s_0} = \mathbf{Dim\ F}$. The packing dimension has many similar properties to the Hausdorff dimension.

Let A, B be Borel sets. Some properties of Packing dimension are:
(1) If $A \subseteq B$, then Dim $A \leq$ Dim B.
(2) $\text{Dim}(A \cup B) = \max \{\text{Dim } A, \text{Dim } B\}$.

As we had before, we may consider a function realizing a ratio. Let $f : S \to T$ be a similarity with ratio $r > 0$, let s be a positive real number, and let $F \subseteq S$ be a set. Then $\overline{\mathcal{P}}^s(f[F]) = r^s \overline{\mathcal{P}}^s(F)$. So Dim $f[F] =$ Dim F.

6.3 Fractal Definition

Now we have done enough leg work to get to a definition for a fractal. There are two definitions, Mandelbrot and Taylor's.

Definition 6.9. *Mandelbrot defined a fractal to be a set A with Cov $A <$ dim A. Taylor defined a fractal saying, a set $F \subseteq \mathbb{R}^d$ is a* **fractal** *if and only if dim $F =$ Dim F.*

Theorem 6.10. *Let S be a metric space and $F \subseteq S$ a Borel set. Then, in general we have,*

$$\mathcal{H}^s(F) \leq 2^s \mathcal{P}^s(F) \text{ and } \dim F \leq \text{ Dim } F$$

Proof 6.11. *We begin by showing that $\overline{\mathcal{H}}^s_{4\epsilon}(F) \leq 2^s \overline{\mathcal{P}}^s_{\epsilon}(F)$. In the first case, if $\overline{\mathcal{P}}^s_{\epsilon}(F) = \infty$, then the equation holds. Suppose then, that $\overline{\mathcal{P}}^s_{\epsilon}(F) < \infty$. This implies that there is a maximal* **finite** *packing, since any infinite packing of F, with all radii equal to ϵ would give an infinite value to the s-dimensional packing measure.*

Call this maximal finite packing of F, $\prod_\epsilon = \{(x_1, \epsilon), (x_2, \epsilon), \ldots, (x_n, \epsilon)\}$. Therefore, by Definition 6.8 we have

$$\overline{\mathcal{P}}^s_{\epsilon}(F) = \sup \sum_{(x,\epsilon) \in \prod_\epsilon} (2\epsilon)^s \geq n(2\epsilon)^s$$

By the maximality, for any $x \in F$, there is some i such that $\rho(x, x_i) \leq 2\epsilon$. Therefore, the collection $\{B_{2\epsilon}(x_i) : 1 \leq i \leq n\}$ covers F, and

$$\overline{\mathcal{H}}^s_{4\epsilon}(F) \leq \sum_{i=1}^{n} (\text{diam } B_{2\epsilon}(x_i))^s \leq n(4\epsilon)^s = 2^s n(2\epsilon)^s \leq 2^s \overline{\mathcal{P}}^s_{\epsilon}(F)$$

As $\epsilon \to 0$ we get that $\mathcal{H}^s(F) \leq 2^s \mathcal{P}^s_0(F)$. □

This leads us to the conclusion that **dim** $F \leq$ **Dim** F. This brings us back to the definition of a fractal, according to Taylor. Generally speaking, dim $F \leq$ Dim F. However, for special sets F we have that dim $F =$ Dim F. These sets are fractals.

Let us quickly apply Mandelbrot's definition to the Sierpinski triangle to see that it is a fractal. Utilizing Remark 6.4 we calculate the Hausdorff dimension to be *dim* $S \approx 1.585$. Meanwhile, we can cover the Sierpinski triangle in a similar fashion as was done in Figure 4; with a family of balls whose order is 1 (any three balls has empty intersection). Therefore, $Cov\ S = 1$ and we conclude, $Cov\ S\ <\ dim\ S$. This confirms that the Sierpinski triangle is indeed a fractal.

7 Fractals and the Complex Plane

One of the beautiful aspects of mathematics is its consistency and symmetry within itself. There are incredibly beautiful patterns, shapes and designs that all have mathematical foundations. In the study of complex analysis and functions in the complex plane, there arise wonderfully intricate shapes and designs. Seen in Figure 6 is the famous Mandelbrot Set and one particular Julia Set.

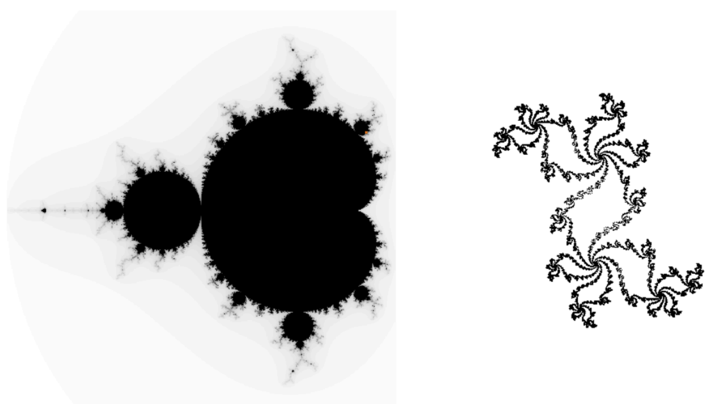


Figure 6: Mandelbrot and a Julia Set

7.1 Julia Sets

As was mentioned in the introduction, Julia Sets come about through an iterated function system. Let $\tau : \mathbb{C} \to \mathbb{C}$ be a function defined by $\tau(z) = z^2 + c$, where c is a fixed complex number. Then one can choose $z \in \mathbb{C}$ and begin an iteration.

Example: Let $c = -0.15 + 0.72i$. Now, take $z = 1 + 2i$. The iteration becomes (numbers rounded to 2 decimal places)

$$\tau_1(1 + 2i) = (1 + 2i)^2 + (-0.15 + 0.72i)$$
$$\tau_2(-3.15 + 4.72i) = (-3.15 + 4.72i)^2 + (-0.15 + 0.72i)$$
$$\tau_3(-12.51 - 29.02i) = (-12.51 - 29.02i)^2 + (-0.15 + 0.72i)$$
$$\vdots$$
$$\tau_n(z_{n-1}) = (z_{n-1})^2 + (-0.15 + 0.72i)$$

This iteration creates a sequence, τ_n, of complex numbers. In this particular case the sequence diverges. However, there may be some complex number, z_0 for which the iterated function produces a convergent sequence. There are even complex numbers for which the iterated function produces a bounded sequence, not necessarily convergent but perhaps oscillating between two finite values.

7.2 Mandelbrot Set

Many of the fractals investigated above were not random. However, many fractals in the complex plane seem to be very random. The Mandelbrot set is the set of all points $c \in \mathbb{C}$ such that, upon iteration, the function $f_c(z) = z^2 + c$, beginning at $z = 0$, remains close to zero. It is impossible to note every element in the set. Even two complex numbers, that are arbitrarily close to one another in the complex plane, may have completely different sequences, one diverging to ∞ while the other point remains close to 0 or converges to 0. In particular, the boundary of this set exhibits complicated structures at all levels.

The mathematical complexities of the Mandelbrot set have yet to be studied in detail. We do know that the Mandelbrot set is connected, though the proof is beyond the scope of this paper. One can also note that the Mandelbrot set has a finite area. Each of the complex numbers that lie in the Mandelbrot set, denoted M, are all within a distance of 2 from the origin. That is, every $z \in M$ lies inside the circle with radius 2, otherwise denoted $|z| \leq 2$. However, this doesn't tell us the area of the set M. In order to calculate the area of the set one would have to run infinitely many iterations, all infinitely long. Therefore, finite estimates of the area is where mathematics is currently at. The circle with radius 2 gives an upper bound area of 12.6. Using the points furthest from the origin and forming a rectangle gives an upper bound area of 5.5.

When we discussed the Sierpinski gasket we concluded that S was more "complicated" than being simply 1-dimensional. There is a measure of roughness to it's boundary that actually makes it 1.58 dimensional. Now, the Mandelbrot set has a very complicated boundary. Therefore, we would assume that the dimension of its boundary was greater than 1. In fact, the boundary is as wiggly, as complicated, and as rough as possible. The dimension of the boundary for the Mandelbrot set is 2.

This is one of the reasons the Mandelbrot set is so fascinating: It encloses a finite area (best approximations around 1.50648) with an infinitely long perimeter. Furthermore, the perimeter is as rough as possible and is not totally self-similar, yet has amazing local similarities and patterns that are incredibly aesthetically pleasing.

8 Conclusion

We have seen 4 different notions of dimension. First, the covering dimension which was an integer value achieved by covering a set F with a family of sets and examining the intersection of $n+2$ of the sets from the family. This gave us the topological dimension of a set. Then, we discovered the similarity dimension, which was easy to compute once a ratio list and iterated function system were defined. The similarity dimension gave fractional values and rightly conveyed the "roughness" of a set, but the severe limitations of similarity and not being well defined for sets only were significant draw backs. Therefore, we defined the Hausdorff and Packing dimensions, using coverings and packings respectively. Both of these notions of dimension gave fractional values, were decreasing step functions of s, and were limiting processing using ϵ-covers and δ-fine packings.

With these four definitions at our disposal we realized that Fractals are sets where there are specific relationships between these notions of dimension: either $Cov\ F < dim\ F$ and/or $dim\ F = Dim\ F$.

In conclusion, fractals are sets with extremely complicated properties. Their boundaries may be infinitely intricate, their volume may be near impossible to calculate, and yet they may look beautifully simple. The study of fractals will continue to be a interesting field of mathematics and its physical applications (Brownian motion, computer graphics, etc) will become more apparent with further study.

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